

# Phase transition for loop covering probabilities

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## Abstract

Given a sequence of undirected connected graphs  $G_n$  of  $m_n$  vertices with bounded degrees and weights, one can define a sequence of transition matrices  $Q_n$  on the graphs  $G_n$ . Given an additional sequence of killing parameters  $c_n$ , one can define a sequence of non-normalized loop measures with killing parameters  $c_n$ . After normalization, one can define a sequence of probabilities  $\mathfrak{P}_n$  on non-trivial loops on  $G_n$ . Assuming that the empirical distributions  $\nu_n$  of the eigenvalues of the transition matrices  $Q_n$  converge, the main result is the determination of the limit of  $\mathfrak{P}_n[\text{loop covers every vertex}]$  as  $n \rightarrow \infty$ . Let  $\mathcal{L}_n$  be the Poisson soup of loops with intensity  $\frac{\mu_n}{m_n}$ . A corollary of the main result is the limit law of the number of the loops covering the whole graph. This paper also gives a criterion for the convergence of the empirical distributions of the eigenvalues of the transition matrices. Besides, we prove that little changes (much smaller than the size of the graph) on the edges of  $G_n$  do not effect the limit of  $\mathfrak{P}_n[\text{loop covers every vertex}]$  as  $n \rightarrow \infty$ . Several examples and counter examples are presented at the end of the paper.

# 1 Introduction

Let  $(G_n = (V_n, E_n, w_n), n \geq 0)$  be a sequence of undirected connected graphs. We assume that  $G_n$  has  $n$  vertices  $V_n$ ,  $E_n$  is the set of edges and  $w_n$  the weights of the edges. From the sequence of graphs  $(G_n, n \geq 0)$ , one obtains a sequence of Markov chains  $(X^{(n)} = (X_1^{(n)}, \dots, X_k^{(n)}, \dots), n \geq 0)$  on the graphs with transition matrices  $Q_n$ , where  $(Q_n)_y^x = \frac{(w_n)_{xy}}{\sum_y (w_n)_{xy}}$ . A pointed loop can be viewed as an element  $(\xi_1, \dots, \xi_k)$  in  $\bigcup_{k \in \mathbb{N}_+} V^k$ , where  $V$  is the state space and  $k$  is called the number of jumps in the pointed loop. Use  $l$  to stand for a pointed loop, and  $p(l)$  to stand for the jumps in the pointed loop. When  $p(l) = 1$ , the pointed loop is considered trivial. The non-trivial pointed loop measure  $\mu_n$  with respect to the transition matrix  $Q_n$  and the additional killing rate  $c_n$  is defined as

$$\mu_n(k \text{ jumps}, \xi_1 = x_1, \dots, \xi_k = x_k) = \frac{1}{k} \left( \frac{1}{1 + c_n} \right)^k (Q_n)_{x_2}^{x_1} (Q_n)_{x_3}^{x_2} \dots (Q_n)_{x_1}^{x_k}$$

Given  $(\xi_1, \dots, \xi_k)$  in  $\bigcup_k V^k$ , it can be extended periodically to a sequence  $(\xi_n, n \in \mathbb{Z}) \in V^{\mathbb{Z}}$  such that  $\xi_{i+kp} = \xi_i$  for  $i = 1, \dots, k$  and  $p \in \mathbb{Z}$ . For  $m \in \mathbb{Z}$  and  $l = (\xi_1, \dots, \xi_k)$ , define the circular translation as  $\theta_m(l) = (\xi_{m+1}, \dots, \xi_{m+k})$ . We call  $l_1$  is equivalent to  $l_2$  when there exists some  $m \in \mathbb{Z}$  such that  $\theta_m(l_1) = l_2$ . The space of loops is defined as the quotient space with respect to the above equivalent relation. The loop measure  $\mu_{n,o}$  (“o” stands for “loop”) is defined as the image measure under this quotient map. The non-trivial loop measures (resp. pointed loop measures)  $\mu_{n,o}$  (resp.  $\mu_n$ ) are finite. After normalization, one gets a sequence of non-trivial loop probabilities  $\mathfrak{P}_n^o$  and non-trivial pointed loop probabilities  $\mathfrak{P}_n$ .

$$\mathfrak{P}_n^o = \frac{\mu_{n,o}}{||\mu_{n,o}||}, \mathfrak{P}_n = \frac{\mu_n}{||\mu_n||}$$

We assume that the degrees and the weights in the graphs are uniformly bounded, and have a lower bound strictly positive. Suppose one knows the limit of the distributions of the eigenvalues of the transition matrices  $Q_n$ . In this paper, We study the limit of  $\mathfrak{P}_n^o(\{\text{loops covering every vertex}\}) = \mathfrak{P}_n(\{\text{pointed loops covering every vertex}\})$ , see Theorem 2.7.

More precisely, suppose  $(G_n = (V_n, E_n, w_n), n \geq 1)$  is a sequence of undirected connected weighted graphs with maximum degrees  $D_n$  and minimum degrees  $d_n$ . Suppose the degrees are uniformly bounded from above and below,  $D_n \leq D < +\infty$  and  $d_n \geq d > 0$  for  $n \geq 1$ .  $V_n$  is the set of vertices and  $E_n$  is the set of edges. Each edge  $\{x, y\}$  is associated with a positive weight  $w_{xy} = w_{yx}$ . Let  $w_x = \sum_y w_{xy}$ ,  $w = \sum_x w_x$  and  $\pi_x = w_x/w$ . Suppose  $1/w_{xy}$  is uniformly bounded from above and below for all graphs, use  $R$  and  $r$  to stand for the upper bound and lower bound. Suppose  $0 < r \leq R < \infty$ . Use  $m_n$  to stand for the numbers of the vertices,  $m_n = |V_n|$ .

For example, take  $V_n = \mathbb{Z}^d / n\mathbb{Z}^d$ . There is a map from  $\mathbb{Z}$  to  $V_n$  which maps the vector  $v$  to  $[v] \in V_n$  (the equivalent class of  $v$ ). The edge set  $E_n$  is defined by  $\{\{[u], [v]\} : u, v \in \mathbb{Z}^d \text{ and the distance between } u \text{ and } v \text{ is } 1\}$ . Finally, give each edge the same weight 1. Then  $w_{xy} = 1$  for all  $\{x, y\} \in E_n$ ,  $w_x = d$ ,  $w = dn$ ,  $\pi = 1/n$ ,  $D_n = d_n = d$  and  $R = r = 1$ . Suppose  $(X_m^{(n)}, m \in \mathbb{N})$  is the Markov chain associated to  $G_n$  with transition matrix  $Q_n((Q_n)_y^x = w_{xy}/w_x$  if and only if  $\{x, y\}$  is an edge of  $G_n$  and  $(Q_n)_y^x = 0$  otherwise). For more details about the loop measure, please refer to Le Jan's book [7].

For the sake of simplicity, use  $\mathcal{C}$  to stand for the event  $\{1 \text{ covers every vertex}\}$ .

The problem is to find reasonable conditions on  $c_n$  such that the limit of  $\mathfrak{P}_n^o(\mathcal{C}) = \mathfrak{P}_n(\mathcal{C})$  exists as  $n \rightarrow \infty$  and can be computed (especially to find conditions under which it is non-trivial, i.e. belongs to  $]0, 1[$ ).

**Assumption:** The distributions of the eigenvalues of the transition matrices  $Q_n$  converge as  $n \rightarrow \infty$ .

The traditional covering problem is about the mean covering time  $C$  at which a random walk on the weighted graphs  $G = (V, E, w)$  has visited every vertex. Often, one considers the covering-and-return time  $C^+$ , which is defined as the first return time to the initial

point after the covering time. Using a spanning tree argument, Theorem 1 in Chapter 6 of the book [1] shows that

$$\mathbb{E}^v(C^+) \leq \sum_{x,y} w_{xy} \min_{T \text{ is a spanning tree}} \sum_{\text{edge } e \in T} \frac{1}{w_e}$$

Moreover, Lemma 25 in Chapter 6 of the book [1] shows that  $\max_v \mathbb{E}^v(C^+), \min_v \mathbb{E}^v(C^+)$  and  $\mathbb{E}^\pi(C)$  are equivalent up to some constants independent of the undirected weighted graphs.

Our problem is quite different from the traditional covering problem. It is mainly about an estimation of the length of the loops, see Proposition 2.5. The proof is based on an upper bound on the transition functions of symmetric Markov processes. It is related to an estimate on Dirichlet forms, proved in [3]. Proposition 18 of Chapter 6 in [1] gives the result for regular case through an elementary argument which is used here to get the estimations for the traces of the transition matrices.

More precisely, as the size of graph grows to infinity, consider the conditional probability of covering the whole space  $V_n$  given the length of the loop. It is zero for length smaller than  $m_n$ , where  $m_n$  is the size of the graph. It tends to 1 for length larger than  $m_n^4$ . In order to prove this, one needs the result of the traditional covering time problem. To be more precise, one needs an upper bound of the expectation of the covering-and-return time, which can be found as Theorem 1 in the Chapter 6 of the book [1]. Its argument goes back to [2]. And the probability that the length is between  $m_n$  and  $m_n^4$  tends to 0 as  $n$  tends to infinity. It is a consequence of an estimate for the traces of the transition matrices.

## 2 The limit of the percentage of those non-trivial loops containing all the vertices

We first suppose  $|V_n| = n$ , as the case  $|V_n| = m_n \neq n$  is almost the same. The difference will appear in the statement of the last theorem.

Write the eigenvalues of  $Q_n$  in non-decreasing order  $-1 \leq \lambda_{n,1} \leq \dots \leq \lambda_{n,n} \leq 1$  (all the eigenvalues are real because the Markov chain is reversible).

**Assumption:** Suppose  $\nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_{n,i}}$  converges in distribution to a probability measure  $\nu$  on  $[-1, 1]$ .

As the trace of  $Q_n$  equals zero,  $\int x \nu(dx) = 0$ .

**Recall** that  $\mathcal{C}$  stands for the event  $\{1 \text{ covers every vertex}\}$ .

## 2.1 The d-regular case

### 2.1.1 Aperiodic case

Suppose all the graphs  $G_n$  are d-regular and aperiodic with weight 1 on every edge.

The following proposition is taken from [5] page 37. It gives an estimation of the second largest eigenvalue of an irreducible Markov transition matrix.

Suppose  $V$  is a finite state space and let  $Q_y^x$  be the transition matrix of an irreducible Markov chain on  $V$ . Assume  $Q$  is reversible with respect to a probability measure  $\pi$ . Set

$$w_y^x = \pi_x Q_y^x$$

Then  $w$  is symmetric. One can define a weighted undirected graph  $G = (V, E, w)$  as following:  $E = \{\{x, y\} : x, y \in V, w_y^x > 0\}$  ( $V$  is the set of vertices,  $E$  is the set of edges and  $w$  is the set of conductances). It is connected since the Markov chain is irreducible.

For any different  $x, y \in V$ , there exists at least a self-avoiding path from  $x$  to  $y$ . Choose one such path arbitrarily and denote it by  $\gamma_{xy}$ . Define the path length  $|\gamma_{xy}|_w$  to be

$\sum_{\{u,v\} \in \gamma_{xy}} 1/w_v^u$ . Define

$$\kappa = \max_{u,v \in V} \sum_{\{x,y \in V : \{u,v\} \in \gamma_{xy}\}} |\gamma_{xy}|_w \pi_x \pi_y$$

Moreover, since  $Q$  is aperiodic, for each  $x \in V$ , there exists at least one path from  $x$  to  $x$  with odd number of edges. Choose one such path, namely  $\sigma_x$ . Define the length of the

path related to  $w$ :

$$|\sigma_x|_w = \sum_{\text{edge } \{u,v\} \in \text{path } \sigma_x} 1/w_v^u$$

Define  $\tau = \max_e \sum_{\{x: \sigma_x \text{ contains edge } e\}} \pi_x |\sigma_x|_w$ .

**Proposition 2.1** (Poincaré inequality). *The second largest eigenvalue  $\beta_1$  satisfies:*

$$\beta_1 \leq 1 - 1/\kappa$$

for any choice of  $\gamma_{xy}(x, y \in V)$ .

The following proposition is Proposition 2 in [5]

**Proposition 2.2.** *Suppose  $Q$  is aperiodic, then the smallest eigenvalue of  $Q$  satisfies*

$$\beta_{\min} \geq -1 + \frac{2}{\tau}$$

for any choice of  $\sigma_x(x \in X)$ .

**Corollary 1.** *a) For a regular connected graph with degree  $d$ , one can choose  $\gamma_{xy}$  such that  $|\gamma_{xy}|_w \leq dn^2$ , which implies that  $\kappa \leq dn^2$ . Therefore, the second largest eigenvalue  $\beta_1$  satisfies:*

$$\beta_1 \leq 1 - \frac{1}{dn^2}$$

*b) For a regular connected aperiodic graph with degree  $d$ , one can choose  $\sigma_x$  in order that  $|\sigma_x|_w \leq 3dn^2$ , which implies that  $\tau \leq 3dn^2$ . Therefore, the smallest eigenvalue  $\beta_{\min}$  satisfies:*

$$\beta_{\min} \geq -1 + \frac{2}{3dn^2}$$

*Proof.* In the Poincaré's inequality,  $\pi$  is the stationary probability measure. Specially, in case of the regular graphs, it is uniformly distributed on vertices and  $w_{xy} = \pi_x Q_y^x = \frac{1}{dn}$ .

For the part a), one could choose  $\gamma_{xy}$  to be self-avoiding and consequently its length is no more than  $n - 1$ . Thus,  $|\gamma_{xy}|_w \leq dn^2$  and

$$\beta_1 \leq 1 - \frac{1}{dn^2}$$

For the part b), among all the loop with odd number of edges, there is a loop with fewest edges, namely  $\sigma$ . Then  $\sigma$  is necessarily self-avoiding. Accordingly, the number of edges in  $\sigma$  is no more than  $n$ . Suppose the loop  $\sigma$  visits  $x_0$ . For any  $x \in V$ , there are a path  $\gamma_{xx_0}$  from  $x$  to  $x_0$  and its reverse  $\gamma_{x_0x}$ . The sum of  $\gamma_{xx_0}$ ,  $\gamma_{x_0x}$  and  $\sigma$  is a loop containing  $x$  with no more than  $3n$  edges. So,  $|\sigma_x|_w \leq 3dn^2$  and  $\tau \leq 3dn^2$ .  $\square$

**Corollary 2.** Fix  $b > 2$ ,  $\lim_{n \rightarrow \infty} \sup_{k \geq n^b} |tr Q_n^k - 1| = 0$ .

*Proof.* By Corollary 1, for any eigenvalue  $\lambda$  different from 1,  $1 - |\lambda| \geq \frac{2}{3dn^2}$ . Since  $Q_n$  is aperiodic, 1 is the only eigenvalue with module 1.

$$\begin{aligned} |tr Q_n^k - 1| &= \left| \sum_{\lambda \text{ is an eigenvalue of } Q_n} \lambda^k - 1 \right| = \left| \sum_{\lambda \text{ is an eigenvalue different from 1}} \lambda^k \right| \\ &\leq \sum_{\lambda \text{ is an eigenvalue different from 1}} |\lambda|^k \leq n(1 - \frac{2}{3dn^2})^k \end{aligned}$$

Consequently,  $\lim_{n \rightarrow \infty} \sup_{k \geq n^b} |tr Q_n^k - 1| = 0$ .  $\square$

**Proposition 2.3.** In the case of a regular connected graph with degree  $d$ ,  $\mathbb{P}^x[X_k = x] \leq 10(k^{-1/2} \vee \frac{1}{n})$  for any  $x$ .

*Proof.* By Proposition 18 of Chapter 6 in [1], for any  $x, y$

$$\mathbb{P}^x[X_k = y] \leq 10k^{-1/2}, \quad k \leq n^2$$

Conditioning with respect to  $X_{k-n^2}$ , we get

$$\mathbb{P}^x[X_k = y] \leq \frac{10}{n}, \quad k \geq n^2$$

$\square$

**Proposition 2.4.** a)  $\mu_n(p(l) \in [n, n^2]) \leq \frac{20n/\sqrt{n-1}-20}{(1+c_n)^n}$

b)  $\mu_n(p(l) \in [n^2, n^b])(b > 2) \leq \frac{10b \ln n}{(1+c_n)^{n^2}}$

c) If  $b > 2$ , then  $\mu_n(p(l) \geq n^b) \sim \sum_{k \geq n^b} \frac{1}{k} (\frac{1}{1+c_n})^k$

d)  $\mu_n(2 \leq p(l) < n)/n \in [\frac{1}{2(1+c_n)^2 d^2}, 20]$

*Proof.* a)b) By Proposition 2.3,

$$\mu_n(p(l) \in [n, n^2]) = \sum_{k=n}^{n^2} \frac{1}{k(1+c_n)^k} \left( \sum_{x \in V_n} \mathbb{P}^x[X_k = x] \right) \leq \left( \frac{1}{1+c_n} \right)^n \sum_{k=n}^{n^2} \frac{10n}{k} k^{-1/2} \leq \frac{20 \frac{n}{\sqrt{n-1}} - 20}{(1+c_n)^n}$$

$$\mu_n(p(l) \in [n^2, n^b]) = \sum_{k=n^2}^{n^b} \frac{1}{k} \left( \sum_{x \in V_n} \mathbb{P}^x[X_k = x] \right) \left( \frac{1}{1+c_n} \right)^k \leq \left( \frac{1}{1+c_n} \right)^{n^2} \sum_{k=n^2}^{n^b} \frac{1}{k} n \frac{10}{n} \leq \frac{10b \ln n}{(1+c_n)^{n^2}}$$

c) By Corollary 2,  $\mu_n(p(l) \geq n^b) (b > 2) = \sum_{k \geq n^b} \frac{1}{k} \text{tr} Q_n^k \left( \frac{1}{1+c_n} \right)^k \sim \sum_{k \geq n^b} \frac{1}{k} \left( \frac{1}{1+c_n} \right)^k$  as  $n \rightarrow \infty$ .

d)

$$\mu_n(2 \leq p(l) < n) = \sum_{k=2}^{n-1} \frac{1}{k} \left( \frac{1}{1+c_n} \right)^k \text{tr} Q_n^k \in \left[ \frac{1}{2} \left( \frac{1}{1+c_n} \right)^2 \text{tr} Q_n^2, \sum_{k=2}^n \frac{1}{k} \text{tr} Q_n^k \right]$$

In the case of the regular graph with degree  $d$ ,  $\text{tr} Q_n^2 \geq \frac{n}{d^2}$ . While by Proposition 2.3,  $\text{tr} Q_n^k \leq \frac{10n}{k^{1/2}}$ . As the consequence,  $\mu_n(2 \leq p(l) < n)/n \in [\frac{1}{2(1+c_n)^2 d^2}, 20]$ .  $\square$

**Proposition 2.5.** a.1) If  $\liminf_{n \rightarrow \infty} c_n > 0$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) < n) = 1$ .

a.2) If  $\lim_{n \rightarrow \infty} c_n = 0$  and  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n} = 0$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) < n) = 1$ .

b) If  $\lim_{n \rightarrow \infty} c_n = 0$  and  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n} = \infty$ ,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) \geq n^4) = 1$ .

c) If  $\lim_{n \rightarrow \infty} c_n = 0$  and  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n} = a \in ]0, \infty[$ ,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) \geq n^4) = \lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) \geq n) = \frac{a}{a + \int -\ln(1-x) \nu(dx)}$ . Besides,  $\int -\ln(1-x) \nu(dx) \in ]0, 20[$

*Proof.* a.1) By the estimation in Proposition 2.4,  $\mu_n(p(l) < n) \geq \frac{n}{2(1+c_n)^2 d^2}$ ,  $\mu_n(p(l) \in [n, n^2]) \leq \frac{20n}{\sqrt{n-1}(1+c_n)^2} = o(\mu_n(p(l) < n))$ ,  $\mu_n(p(l) \in [n^2, n^4]) \leq \frac{40 \ln n}{(1+c_n)^2} = o(\mu_n(p(l) < n))$  and  $\mu_n(p(l) > n^4) \leq \sum_{k \geq 1} \frac{1}{k} \left( \frac{1}{1+c_n} \right)^k = -\ln(c_n) + \ln(1+c_n) = o(\mu_n(p(l) < n))$ . In short,

$\mu_n(p(l) \geq n) = o(\mu_n(p(l) < n))$ . So,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) < n) = \lim_{n \rightarrow \infty} \frac{\mu_n(p(l) < n)}{\mu_n(p(l) < n) + \mu_n(p(l) \geq n)} = 1$

b) In this case,  $\lim_{n \rightarrow \infty} \sup_{k \leq n^4} |(1+c_n)^k - 1| = 0$ . By Proposition 2.4,  $\mu_n(p(l) \geq n^4) \sim$

$$\sum_{k \geq n^4} \frac{1}{k} \left( \frac{1}{1+c_n} \right)^k = \sum_{k \geq 1} \frac{1}{k} \left( \frac{1}{1+c_n} \right)^k - \sum_{k < n^4} \frac{1}{k} \left( \frac{1}{1+c_n} \right)^k = -\ln(c_n) + \ln(1+c_n) - \sum_{k < n^4} \frac{1}{k} \left( \frac{1}{1+c_n} \right)^k.$$

Since  $|\sum_{k < n^4} \frac{1}{k} \left( \frac{1}{1+c_n} \right)^k| \sim \sum_{k < n^4} \frac{1}{k} \sim 4 \ln n$ ,  $\mu_n(p(l) \geq n^4) \sim -\ln(c_n)$ . By Proposition 2.4,

$\mu_n(p(l) < n^4) = O(n) = o(-\ln(c_n)) = o(\mu_n(p(l) \geq n^4))$ . Therefore,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) \geq n^4) =$

$$\lim_{n \rightarrow \infty} \frac{\mu_n(p(l) \geq n^4)}{\mu_n(p(l) \geq n^4) + \mu_n(p(l) < n^4)} = 1.$$



c) If  $\lim_{n \rightarrow \infty} c_n = 0$  and  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n} = a \in ]0, \infty[$ , then  $\lim_{n \rightarrow \infty} \sup_{k \leq n^4} |(\frac{1}{1+c_n})^k - 1| = 0$ .

For  $k$  fixed, as  $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_{n,i}}$  converges in distribution to the probability measure  $\nu$ ,

$$\lim_{n \rightarrow \infty} \frac{\text{tr} Q_n^k}{n} = \lim_{n \rightarrow \infty} \frac{\lambda_{n,1}^k + \cdots + \lambda_{n,n}^k}{n} = \int x^k \nu(dx)$$

Note that

$$\begin{aligned} \lim_{A \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu_n(p(l) \in [A, n])/n &= \lim_{A \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k=A}^n \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \frac{\text{tr} Q_n^k}{n} \\ &\leq \lim_{A \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k=A}^n \frac{1}{k} \frac{\text{tr} Q_n^k}{n} \end{aligned}$$

By Proposition 2.3,  $\frac{\text{tr} Q_n^k}{n} \leq \frac{10}{k^{1/2}}$  for  $k \leq n^2$ . Therefore,

$$\lim_{A \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu_n(p(l) \in [A, n])/n \leq \lim_{A \rightarrow \infty} \sum_A^{\infty} \frac{10}{k^{3/2}} = 0$$

Consequently,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\mu_n(p(l) \in [2, n])}{n} &= \lim_{A \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\mu_n(p(l) \in [2, A])}{n} = \lim_{A \rightarrow \infty} \sum_{k=2}^A \frac{1}{k} \int x^k \nu(dx) \\ &= \int (-\ln(1-x) - x) \nu(dx) = \int -\ln(1-x) \nu(dx) \end{aligned}$$

Since  $\frac{1}{4d^2} < \frac{\mu_n(p(l) \in [2, n])}{n} < 20$  for  $n$  large,  $\int -\ln(1-x) \nu(dx) \in ]0, 20[$ .

$$\begin{aligned} \mu_n(p(l) \geq n^4) &\sim \sum_{k \geq n^4} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k = \sum_{k \geq 1} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k - \sum_{k=1}^{n^4-1} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \\ &= -\ln c_n + \ln(1+c_n) - \sum_{k=1}^{n^4-1} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \sim -an \end{aligned}$$

And  $\mu_n(p(l) \in [n, n^4]) = o(n)$ . Therefore,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) \geq n^4) = \lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) \geq n) = \frac{a}{a + \int -\ln(1-x) \nu(dx)}$ .  $\square$

**Proposition 2.6.** Take  $\epsilon > 0$ ,  $\lim_{n \rightarrow \infty} \sup_{k \geq n^{3+\epsilon}} |\mathfrak{P}_n(\mathcal{C} | p(l) = k) - 1| = 0$ .

*Proof.* According to Theorem 1 in the Chapter 6 of the book [1], the expectation of the “cover-and-return” time is bounded from above by  $dn(n-1)$ . To be more precise, let  $T_{n,y} = \inf\{m \geq 0, X_m^{(n)} = y\}$ . Define the covering time by  $C_n = \max_{y \in V_n} T_{n,y}$ . Define the cover-and-return time  $C_n^+ = \inf\{m \geq 1 : m \geq C_n, X_m^{(n)} = X_0^{(n)}\}$ . Then  $\max_x \mathbb{E}_n^x[C_n^+] \leq dn(n-1)$ . By Markov’s inequality,  $\max_x \mathbb{P}_n^x[C_n^+ \geq n^{3+\epsilon}] \leq \frac{dn(n-1)}{n^{3+\epsilon}}$ .

$$\begin{aligned}
& \mathfrak{P}_n(1 \text{ does not cover every vertex} \mid p(l) = k) \\
&= \frac{\mathfrak{P}_n(1 \text{ does not cover every vertex}, p(l) = k)}{\mathfrak{P}_n(p(l) = k)} \\
&= \frac{\mu_n(1 \text{ does not cover every vertex}, p(l) = k)}{\mu_n(p(l) = k)} \\
&= \frac{\frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \sum_{x \in V_n} \mathbb{P}^x[X_k = x, X \text{ does not cover every vertex before time } k]}{\frac{1}{k} \left(\frac{1}{1+c_n}\right)^k \text{tr} Q_n^k} \\
&\leq \frac{\sum_{x \in V_n} \mathbb{P}^x[C_n^+ > k]}{\text{tr} Q_n^k} \leq \frac{d}{n^\epsilon \text{tr} Q_n^k}
\end{aligned}$$

By Corollary 2,  $\text{tr} Q_n^k$  tends to 1 uniformly for  $k \geq n^{3+\epsilon}$ .

Therefore,  $\lim_{n \rightarrow \infty} \sup_{k \geq n^{3+\epsilon}} |\mathfrak{P}_n(\mathcal{C} \mid p(l) = k) - 1| = 0$ . □

Finally, one can conclude that:

**Theorem 2.7.** *a.1) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n} \leq 0$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 0$ .*

*b) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n} = +\infty$ ,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 1$ .*

*c) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n} = a \in ]0, \infty[$ ,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = \frac{a}{a + \int -\ln(1-x) \nu(dx)}$  ( $\int -\ln(1-x) \nu(dx) \in ]0, 20[$ ).*

*In the case  $|V_n| = m_n$ ,  $\frac{-\ln c_n}{n}$  should be replaced by  $\frac{-\ln c_n}{m_n}$*

**Corollary 3.** *Let  $\mathcal{L}_n$  be the Poisson collection of loops with intensity  $\frac{\mu_n}{n}$ . Suppose  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n} = a$ . Then,  $\sum_{l \in \mathcal{L}_n} 1_{\{l \in \mathcal{C}\}}$  converges in distribution to a Poisson random variable with parameter  $a \vee 0$  as  $n$  tends to infinity. In the case  $|V_n| = m_n$ ,  $\frac{-\ln c_n}{n}$  should be replaced by  $\frac{-\ln c_n}{m_n}$ .*

*Proof.* The argument in Proposition 2.5, actually gives the following result:  $\lim_{n \rightarrow \infty} \mu_n(p(l) > n^4)/n = \lim_{n \rightarrow \infty} \mu_n(p(l) \geq n)/n = a \vee 0$ . By Proposition 2.6,  $\lim_{n \rightarrow \infty} \frac{\mu_n(\mathcal{C})}{n} = a \vee 0$ . Therefore,

$\sum_{l \in \mathcal{L}_n} 1_{\{l \in C\}}$  converges in distribution to a Poisson random variable with parameter  $a \vee 0$  as  $n$  tends to infinity.  $\square$

### 2.1.2 Periodic case

If the process is not aperiodic, then the period must be 2 (since  $\text{tr} Q_n^2 > 0$ ). The largest eigenvalue of  $Q_n$  is 1 and the smallest one is -1. In this case, one can divide the vertices into two parts as following: fix a vertex  $x$ , let  $A_n = \{y \in V_n : \sum_{k \geq 0} \mathbb{P}_n^x[X_{2k} = y] > 0\}$  and let  $B_n = V_n - A_n$ . This partition  $\{A, B\}$  does not depend on the choice of  $x$ . Moreover, if the initial distribution is supported on  $A$ (or  $B$ ), then the process  $(X_{2m}^{(n)}, m \in \mathbb{N})$  is an aperiodic Markov process on  $A_n$ (or  $B_n$ ) with transition matrix  $Q_n^2|_{A_n}$ (or  $Q_n^2|_{B_n}$ ). A direct consequence is that  $Q_n^2|_{A_n \times B_n}$  and  $Q_n^2|_{B_n \times A_n}$  are zero matrices. If one puts the eigenvalues of  $Q_n^2|_{A_n}$  and  $Q_n^2|_{B_n}$  together, one gets exactly the square of the eigenvalues of  $Q_n$ .  $1 - |\lambda| \geq \frac{1-|\lambda|^2}{2}$  Therefore, one can still get a similar spectrum gap proposition as Corollary 1 by considering the aperiodic Markov processes on  $A_n$  and  $B_n$ .

For Corollary 2,  $\lim_{n \rightarrow \infty} \sup_{k \geq n^b \text{ and } k \text{ is even}} |\text{tr} Q_n^{2k} - 2| = 0$ . And for  $k$  odd, one has  $\text{tr} Q_n^k = 0$ .

Therefore, the Proposition 2.6 should be modified, one considers only the loops with even length. And the proof remains the same.

Although there is very small change in the calculation, the statement of the Proposition 2.5, the Theorem 2.7 and Corollary 3 is unchanged.

## 2.2 Non-regular case, with unit weights

In this part, suppose all the weights are 1 but the graph is not necessarily  $d$ -regular.

Assume it is aperiodic (Otherwise, it can be treated in the same way as in the last section).

Corollary 1, Corollary 2 and Proposition 2.6 remain valid except that one should replace  $d$  by the maximum degree  $D$  everywhere.

The only problem is Proposition 2.3. Instead, one has the following proposition,

**Proposition 2.8.** *Given any unweighted (all the weights are 1) graph with  $n$  vertices, maximum degree  $D$  and minimum degree  $d$ , define the transition matrix  $(Q_{ij}, i, j = 1, \dots, n)$*

as following: if  $\{i, j\}$  is an edge in the graph, then  $Q_{ij} = \frac{1}{\text{degree of } i}$ ; otherwise,  $Q_{ij} = 0$ . Then, there is a Markov chain  $(X_k, k \geq 0)$  on the graph with transition matrix  $Q$ . One has  $\sum_x \mathbb{P}^x[X_k = x] \leq \frac{14D^2}{d^2} \max(\frac{n}{\sqrt{k}}, 1)$ , for  $k > 1$ .

*Proof.* One needs to generalize Proposition 18 of Chapter 6 in [1]. The proof in [1] still works and it is repeated here, for self containedness.

Use  $N_i(A^c)$  to stand for the number of times the chain visits the vertex  $i$  before hitting  $A^c$ .

1) Suppose  $0 < |A| < n$ ,  $\mathbb{E}^i[N_i(A^c)] \leq 5D|A|/d$

$$\mathbb{E}^i[N_i(A^c)] = \mathbb{P}^i[T_{A^c} < T_i^+] = d(i)r(i, A^c)$$

$T_{A^c}$  is the hitting time for  $A^c$  and  $T_i^+$  is the first return time for  $i$ .  $d_i \leq D$  is the degree of  $i$ , and  $r(i, A^c)$  is the effective resistance between  $i$  and  $A^c$  which is bounded from above by  $5|A|/d$ . For the definition of the effective resistance and the relation between electrical network and reversible Markov chain, please refer to [9].

In order to show  $r(i, A^c) \leq 5|A|/d$ , let us choose one shortest path from  $i$  to  $A^c$ , namely  $i = i_1, \dots, i_{k+1}$  such that  $i_1, \dots, i_k \in A$  and  $i_{k+1} \in A^c$ . If  $k = 1$ ,  $r(i, A^c) \leq 1$ . For  $k > 1$ , consider  $B$  the subset of  $A$  which consists of the vertices adjoint to some  $\{i_j : j = 1, \dots, k-1\}$ ,  $B = \{y \in A : \exists 1 \leq j \leq k-1, y \text{ is adjoint to } i_j\}$ . For each  $b \in B$ ,  $b$  has at most 3 neighbours in  $\{i_1, \dots, i_{k+1}\}$ . Otherwise, one could find a path from  $i$  to  $A^c$  containing  $b$  which is shorter than the path  $i_1, \dots, i_{k+1}$ . Therefore,

$$|B| \geq \frac{\sum_{j=1}^{k-1} \sum_{y \in B} 1_{\{y \text{ is adjoint to } i_j\}}}{3} \geq d(k-1)/3$$

Moreover,  $B$  is contained in  $A$  and hence  $|A| \geq d(k-1)/3$ . It implies that  $r(i, A^c) \leq k \leq \frac{3|A|}{d} + 1 \leq \frac{5|A|}{d}$  if  $d \leq 2|A|$ . If  $d > 2|A|$ , then there exists at least  $d - |A|$  edges from  $i$  to  $A^c$ . Thus,  $r(i, A^c) \leq \frac{1}{d-|A|} \leq \frac{2}{d} \leq \frac{5|A|}{d}$ .

2)  $\sum_i \sum_{t \leq k} \mathbb{P}^i[X_t = i] \leq 1 + 6\frac{D^2}{d^2} \max(\frac{k}{n}, \sqrt{k})$

Let  $\pi$  stand for the invariant probability of the Markov chain and  $d(j)$  is the degree of

vertex  $j$ . Then  $\pi_i = \frac{d(i)}{\sum_j d(j)} \in [\frac{d}{Dn}, \frac{D}{dn}]$ . For that reason,

$$\sum_j \frac{d}{Dn} \mathbb{P}^j[X_k = i] \leq \mathbb{P}^\pi[X_k = i] = \pi_i \leq \frac{D}{dn}$$

Therefore,  $\sum_j \mathbb{P}^j[X_k = i] \leq \frac{D^2}{d^2}$ . Let  $A = \{i : \sum_{t \leq k} \mathbb{P}^j[X_t = i] > s\}$ . Then  $|A| < \frac{k}{s} \frac{D^2}{d^2}$ . Considering the hitting time  $T_{A^c}$ ,

$$\sum_{t \leq k} \mathbb{P}^i[X_t = i] \leq s + \mathbb{E}^i[N_i(A^c)] \leq s + \frac{5k}{s} \frac{D^3}{d^3} \text{ (if } |A| < n)$$

Take  $s = \lfloor \frac{\sqrt{k} D^2}{d^2} \rfloor + 1$  ( $\lfloor x \rfloor$  is the largest integer no more than  $x$ ). Then for  $k \leq n^2$ ,  $|A| < \frac{k D^2}{s d^2} \leq n$  and

$$\sum_{t \leq k} \mathbb{P}^i[X_t = i] \leq \sqrt{k} \left( \frac{D^2}{d^2} + 5 \frac{D}{d} \right) + 1 \leq 1 + 6\sqrt{k} \frac{D^2}{d^2}$$

For  $k > n^2$ , take  $s = \lfloor \frac{k D^2}{n d^2} \rfloor + 1$ , then  $|A| < n$  and

$$\sum_{t \leq k} \mathbb{P}^i[X_t = i] \leq \frac{k D^2}{n d^2} + 1 + 5n \frac{D}{d} < 1 + 6 \frac{D^2}{d^2} \frac{k}{n}$$

Finally,

$$\sum_{t \leq k} \mathbb{P}^i[X_t = i] \leq 1 + 6 \frac{D^2}{d^2} \max\left(\frac{k}{n}, \sqrt{k}\right)$$

$$3) \sum_i \mathbb{P}^i[X_k = i] \leq \frac{14D^2/d^2}{\sqrt{k}}, k > 1$$

Let  $Q$  stand for the transition matrix and  $1 = \lambda_1 \geq \dots \lambda_n \geq -1$  are the eigenvalues of  $Q$ . Then,  $\sum_i \mathbb{P}^i[X_{2k} = i] + \sum_i \mathbb{P}^i[X_{2k+1} = i] = \text{tr} Q^{2k} + \text{tr} Q^{2k+1} = \sum_i \lambda_i^{2k} (1 + \lambda_i)$ , it decreases with  $k$  increasing. Therefore,  $\sum_i \mathbb{P}^i[X_{2k} = i] + \mathbb{P}^i[X_{2k+1} = i] \leq \frac{1}{k+1} \sum_i \sum_{t \leq 2k+1} \mathbb{P}^i[X_t = i] \leq \frac{n}{k+1} (1 + 6 \frac{D^2}{d^2} \max(\frac{2k+1}{n}, \sqrt{2k+1}))$ . Finally,  $\sum_i \mathbb{P}^i[X_k = i] \leq \frac{14D^2}{d^2} \max(\frac{n}{\sqrt{k}}, 1), k > 1$ .  $\square$

Finally, Theorem 2.7 and Corollary 3 is unchanged.

## 2.3 The general case

Compared with the last case, the proof is exactly the same with a few changes in the coefficients. Corollary 2 and its proof remains the same both in the aperiodic case and in the 2-period case.

For Proposition 2.3, replace  $D$  by  $D/r$  and  $d$  by  $d/R$ . The proof is similar.

For Proposition 2.4, there are a few changes in the coefficients:

- Proposition 2.9.** a)  $\mu_n(p(l) \in [n, n^2]) \leq \frac{D^2 R^2}{d^2 r^2} \frac{28n/\sqrt{n-1}-28}{(1+c_n)^n}$   
b) For  $b > 2$  fixed,  $\mu_n(p(l) \in [n^2, n^b]) \leq \frac{14D^2 R^2}{d^2 r^2} \frac{b \ln n}{(1+c_n)^{n^2}}$   
c) For  $b > 2$  fixed,  $\mu_n(p(l) \geq n^b) \sim \sum_{k \geq n^b} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k$   
d)  $\mu_n(2 \leq p(l) < n)/n \in [\frac{r^2}{2(1+c_n)^2 R^2 D^2}, 28 \frac{D^2 R^2}{d^2 r^2}]$

For Proposition 2.5, just replace  $\int -\ln(1-x)\nu(dx) \in ]0, 20[$  by  $\int -\ln(1-x)\nu(dx) \in [\frac{r^2}{R^2 D^2}, \frac{28D^2 R^2}{d^2 r^2}]$ .

For Proposition 2.6, one should use the estimation of the expectation of the covering-and-return time for general reversible Markov process, please refer to Theorem 1 in the Chapter 6 of the book [1].

Finally, for the Theorem 2.7 and Corollary 3, nothing needs to be changed.

### 3 A stability result

**Proposition 3.1.** Assume  $Q_n$  is a  $m_n \times m_n$  transition matrix, the following two statement is equivalent:

- a) The empirical distributions  $\nu_n$  of the eigenvalues of the transition matrices  $Q_n$  converges to  $\nu$  as  $n \rightarrow \infty$ .  
b) For all  $0 < \rho < 1$ ,  $\frac{-\ln \det(1-\rho Q_n)}{m_n}$  converges as  $n \rightarrow \infty$ .

*Proof.* a) $\implies$ b):

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{-\ln \det(1-\rho Q_n)}{m_n} &= \lim_{n \rightarrow \infty} \sum_1^\infty \frac{\text{tr}(Q_n^k) \rho^k}{m_n k} = \sum_1^\infty \lim_{n \rightarrow \infty} \frac{\text{tr}(Q_n^k) \rho^k}{m_n k} \\ &= \sum_1^\infty \int \frac{x^k \rho^k}{k} \nu(dx) = \int -\ln(1-\rho x) \nu(dx) \end{aligned}$$

b) $\implies$ a):

The distributions of the eigenvalues of the transition matrices  $Q_n$  is a tight sequence of probability measures on  $[-1, 1]$ . In order to show its convergence, it is enough to show

that the limits are the same for all convergent subsequence. Finally, for two probabilities  $\nu$  and  $\tilde{\nu}$  on  $[-1,1]$ , by comparing the derivatives of the two parts with respect to  $\rho$ , it can be showed that  $\int -\ln(1 - \rho x)\nu(dx) = \int -\ln(1 - \rho x)\tilde{\nu}(dx)$  for all  $\rho \in [0, 1[$  implies  $\int x^k \nu(dx) = \int x^k \tilde{\nu}(dx)$  for all  $k \geq 1$ . Therefore,  $\nu = \tilde{\nu}$ .  $\square$

**Recall** (Cauchy's interlacing theorem). *Let  $S$  be a  $n \times n$  Hilbert matrix, i.e.  $\bar{S}^t = S$ . Let  $\pi : \{1, \dots, n\} \rightarrow \mathbb{R}^+$  strictly positive and  $M_\pi$  be the diagonal matrix such that  $(M_\pi)_i^i = \pi(i)$ . Let  $A = M_\pi S$ . For each principal minor of  $A$ , its eigenvalues are real. Let  $F \subset \{1, \dots, n\}$ ,  $|F| = m$ . Let  $B = A|_{F \times F}$ . If the eigenvalues of  $A$  are  $\alpha_1 \leq \dots \leq \alpha_n$ , and those of  $B$  are  $\beta_1 \leq \dots \leq \beta_m$ , then for all  $j = 1, \dots, m$ ,*

$$\alpha_j \leq \beta_j \leq \alpha_{n-m+j}$$

**Proposition 3.2.** *Suppose  $(G_n = (V_n, E_n, w_n), n \geq 1)$  is a sequence of undirected weighted graphs. We assume  $G_n$  has  $m_n$  vertices  $V_n$ ,  $E_n$  is the set of edges and  $w_n$  the weights of the edges (if  $e$  is not an edge in the graph, we put  $w_e = 0$ ). Let  $\omega_n$  be a measure on  $V_n$  defined by  $(\omega_n)_x = \sum_y (w_n)_{xy}$ . Let  $(Q_n)_y^x = \begin{cases} (w_n)_{xy}/(\omega_n)_x & \text{if } \{x, y\} \in E_n \\ 0 & \text{otherwise} \end{cases}$ . Suppose  $(G'_n = (V'_n, E'_n, w'_n), n \geq 1)$  is a sequence of undirected weighted sub-graphs such that  $V'_n \subset V_n, E'_n \subset E_n, W'_n = W_n|_{E'_n}$  for all  $n$ . Let  $a_n = |E_n| - |E'_n|$ . Write the eigenvalues of  $Q_n(Q'_n)$  in non-decreasing order  $\lambda_{n,1} \leq \dots \leq \lambda_{n,m_n} (\lambda'_{n,1} \leq \dots \leq \lambda'_{n,m'_n})$ . Let  $\nu_n = \frac{\delta_{\lambda_{n,1}} + \dots + \delta_{\lambda_{n,m_n}}}{m_n} (\nu'_n = \frac{\delta_{\lambda'_{n,1}} + \dots + \delta_{\lambda'_{n,m'_n}}}{m'_n})$ .*

*Suppose:  $a_n = o(m_n)$*

*Then  $\lim_{n \rightarrow \infty} \nu_n = \nu \Leftrightarrow \lim_{n \rightarrow \infty} \nu'_n = \nu$ .*

*Proof.* Since  $a_n = o(m_n)$ ,  $\lim_{n \rightarrow \infty} \frac{m_n}{m'_n} = 1$ .

By Proposition 3.1, it is enough to prove that for all  $\rho > 0$

$$\lim_{n \rightarrow \infty} \frac{\ln \det(1 - \rho Q_n) - \ln \det(1 - \rho Q'_n)}{m_n} = 0$$

Since  $a_n = o(m_n)$ , for  $n$  large enough, there exists  $A_n \subset V_n$  such that  $Q_n|_{A_n} = Q'_n|_{A_n}$  and  $|V_n| - |A_n| \leq a_n$ . It is enough to show that  $\lim_{n \rightarrow \infty} \frac{\ln \det(1 - \rho Q_n) - \ln \det(1 - \rho Q_n|_{A_n})}{m_n} = 0$  and

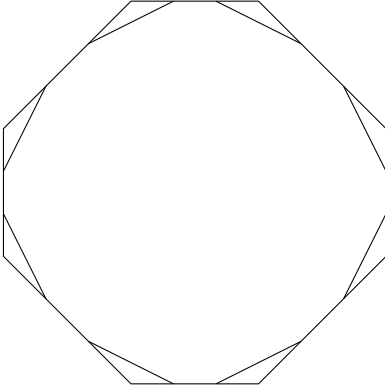
$\lim_{n \rightarrow \infty} \frac{\ln \det(1 - \rho Q'_n) - \ln \det(1 - \rho Q'_n|_{A_n})}{m_n} = 0$ . As following, we will give the proof for the first limit as the second is exactly the same. Let  $\beta_1 \leq \dots \leq \beta_{|A_n|}$  be the eigenvalues of  $Q_n|_{A_n}$ . By Cauchy's interlacing theorem and the fact that  $-1 \leq \lambda_{n,1} \leq \dots \leq \lambda_{n,m_n} \leq 1$ ,

$$\begin{aligned} & \ln \det(1 - \rho Q_n) - \ln \det(1 - \rho Q_n|_{A_n}) \\ &= \ln(1 - \rho \lambda_{n,1}) + \dots + \ln(1 - \rho \lambda_{n,m_n}) - \ln(1 - \rho \beta_1) - \dots - \ln(1 - \rho \beta_{|A_n|}) \\ & \in [\ln(1 - \rho) a_n, \ln(1 + \rho) a_n] \end{aligned}$$

Since  $a_n = o(m_n)$ ,  $\lim_{n \rightarrow \infty} \frac{\ln \det(1 - \rho Q_n) - \ln \det(1 - \rho Q_n|_{A_n})}{m_n} = 0$ .  $\square$

*Remark.* If  $\liminf_{n \rightarrow \infty} \frac{a_n}{m_n} > 0$  and  $m_n \sim m'_n$ , we have the following counter example:

Let  $(G_n = (V_n, E_n, w_n), n \geq 2)$  be a sequence of graphes with equal edge weight 1. Here,  $V_n = \{1, \dots, 3n\}$ ,  $E_{n,1} = \{\{1, 2\}, \{2, 3\}, \dots, \{3n-1, 3n\}, \{3n, 1\}\}$  and  $E_{n,2} = \{\{1, 3\}, \{4, 6\}, \dots, \{3n-2, 3n\}\}$ . Take  $E_n = E_1 \cup E_2$ . The following is the example of  $G_8$ :



Let  $(G'_n = (V'_n, E'_n, w'_n), n \geq 2)$  be another sequence of graphes with equal edge weight 1 such that  $V'_n = V_n$  and  $E'_n = E_{n,1}$ .

By Proposition 2.2, be careful with the definition of  $w$  there,  $\lambda_{n,1} \geq -\frac{11}{12}$ : (Choose  $\sigma_{3i-2} = \sigma_{3i-1} = \sigma_{3i}$  to be the cycle  $3i-2 \rightarrow 3i-1 \rightarrow 3i \rightarrow 3i-2$  for  $i = 1, \dots, n$ .  $(\pi_n)_{3i-1} = \frac{1}{4n}$ ,  $(\pi_n)_{3i-2} = (\pi_n)_{3i} = \frac{3}{8n}$  for  $i = 1, \dots, n$ .  $|\sigma_x|_{w_n} = 24n$  and  $\tau = 24$ . Therefore,  $\lambda_{n,1} \geq -\frac{11}{12}$ .) If  $\nu = \lim_{n \rightarrow \infty} \nu_n$ , then  $\nu[-1, -\frac{11}{12}] = 0$ . While for  $(G'_n, n \geq 2)$ ,  $\lim_{n \rightarrow \infty} \nu'_n$  exists and  $\nu'_n(dy) = 1_{\{y \in [-1, 1]\}} \frac{1}{\pi \sqrt{1-y^2}} dy$ .



## 4 Example: Random walk on the torus

Let  $V_n$  be the discrete torus  $\mathbb{Z}^d/n\mathbb{Z}^d$ . There is a map from  $\mathbb{Z}$  to  $V_n$  which maps the vector  $v$  to  $[v] \in V_n$  (the equivalent class of  $v$ ). The edge set  $E_n$  is defined by  $\{[u], [v]\} : u, v \in \mathbb{Z} \text{ and the distance between } u \text{ and } v \text{ is } 1\}$ . Finally, give each edge the same weight 1. Let  $Q_{d,n}$  be the transition matrix. We will find the limit distribution of the eigenvalues as following.

Let  $(P_{d,n,t}, t \geq 0)$  stand for the semi-group of a simple random walk on  $\mathbb{Z}^d/n\mathbb{Z}^d$  with jumping rate  $d$ . Then  $P_{d,n,t} = e^{-dt(I-Q_{d,n})}$ . Since it can be viewed as  $d$  independent simple random walk on  $\mathbb{Z}^d/n\mathbb{Z}^d$  with jumping rate 1,  $P_{d,n,t} = P_{1,n,t}^{\otimes d}$ . As  $-\frac{dP_{d,n,t}}{dt} = d(I - Q_{d,n})$ ,  $Q_{d,n} = \frac{1}{d}(Q_{1,n} \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes Q_{1,n})$ . For  $d=1$ , the eigenvalues of  $Q_{d,n}$  are  $\cos(\frac{2\pi}{n}p_1)$  for  $p_1 = 0, \dots, n-1$ . Therefore, in general, the eigenvalues of  $Q_{d,n}$  are  $\frac{1}{d}(\cos(\frac{2\pi}{n}p_1) + \cdots + \cos(\frac{2\pi}{n}p_d))$  for  $p_1, \dots, p_d = 0, \dots, n-1$ . Rewrite the eigenvalues in non-decreasing order  $\lambda_1, \dots, \lambda_{n^d}$ . Define  $\nu_n^d = \frac{1}{n^d} \sum \delta_{\lambda_i}$ . Define  $\tilde{\nu}_n^d = \frac{1}{n^d} \sum \delta_{d\lambda_i}$ . Then  $\tilde{\nu}_n^d = (\nu_n^1)^{*d}$  ( $*$  stands for the convolution). For  $f \in C([-1, 1])$ ,

$$\lim_{n \rightarrow \infty} \int f(x) \nu_n^1(dx) = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(\cos(\frac{2\pi}{n}i))/n = \int_0^1 f(\cos(2\pi x)) dx = \int_{]-1,1[} f(y) \frac{1}{\pi \sqrt{1-y^2}} dy$$

Therefore,  $\nu_n^1 \rightarrow 1_{\{y \in ]-1,1[ \}} \frac{1}{\pi \sqrt{1-y^2}} dy$  as  $n \rightarrow \infty$ .

Consequently, let  $m(dy) = 1_{\{y \in [-1/d, 1/d] \}} \frac{d}{\pi \sqrt{1-d^2 y^2}} dy$ , then  $\nu = m^{*d}$ .

As the same argument as the case  $d=1$ ,

$$\int -\ln(1-x) \nu(dx) = \int_{[0,1]^d} -\ln(1 - \frac{\cos(2\pi x_1) + \cdots + \cos(2\pi x_d)}{d}) dx^1 \cdots dx^d$$

Finally, the Theorem 2.7 can be stated more precisely as follows:

- a.1) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n^d} \leq 0$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 0$ .
- b) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n^d} = \infty$ ,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 1$ .
- c) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{n^d} = a \in ]0, \infty[$ ,

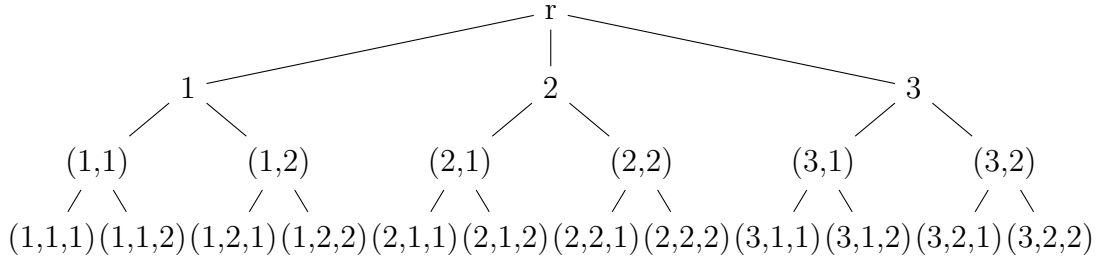
$$\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = \frac{a}{a + \int_{[0,1]^d} -\ln(1 - \frac{\cos(2\pi x_1) + \cdots + \cos(2\pi x_d)}{d}) dx^1 \cdots dx^d}$$

In particular, for  $d = 1$ , the limit equals  $\frac{a}{a + \ln 2}$ .

## 5 Example: Finite quasi-d-regular tree with depth n

**Definition 5.1.** We define the quasi-d-regular tree  $G_n = \{V_n, E_n, w_n\}$  with depth n by recurrence. For all n,  $w_n$  gives equal weight 1 for every edge in  $E_n$ . Define  $V_0 = \{r\}$ ,  $E_0 = \phi$ . Define  $V_1 = \{r, 1, \dots, d\}$ ,  $E_1 = \{\{r, 1\}, \dots, \{r, d\}\}$ . Once  $G_n$  is well-defined for  $n \leq k$ , define  $G_{k+1}$  as following:  $V_{k+1} = V_k \cup ((V_k \setminus V_{k-1}) \times \{1, \dots, d-1\})$  and  $E_{k+1} = E_k \cup \{\{v, (v, j)\} : v \in V_k \setminus V_{k-1}, j \in \{1, \dots, d-1\}\}$ .

The following is an example for  $d = 3, n = 3$ .



Let  $m_n = |V_n|$ , then  $m_0 = 1$  and  $m_n = 1 + d(d-1)^0 + \dots + d(d-1)^{n-1} = \begin{cases} \frac{d(d-1)^n - 1}{d-2} & d > 2 \\ 1 + 2n & d = 2 \end{cases}$ .

Given any matrix  $A$ , let  $A^t$  be the transpose of  $A$ . For  $k, j \in \mathbb{Z}_+$ , let  $I_k$  be the  $k \times k$  identity matrix,  $E_{j,k}$  be the  $j \times k$  matrix such that every element of  $E_{i,k}$  is 1 and  $B_{d,k}$  be a  $d(d-1)^{k-1} \times d(d-1)^k$  matrix defined as following:

$$\begin{pmatrix} E_{1,d-1} & 0 & \cdots & 0 \\ 0 & E_{1,d-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & E_{1,d-1} \end{pmatrix}$$

Let  $Q_{d,n}$  be the transition matrix for the graph  $G_n$ . Then, for  $n \geq 2$ ,

$$Q_{d,n+1} = \begin{pmatrix} 0 & \frac{1}{d}E_{1,d} & 0 & \cdots & \cdots & 0 \\ \frac{1}{d}E_{1,d}^t & 0 & \frac{1}{d}B_{d,1} & \ddots & \ddots & \vdots \\ 0 & \frac{1}{d}B_{d,1}^t & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{d}B_{d,n-1} & 0 \\ \vdots & \ddots & \ddots & \frac{1}{d}B_{d,n-1}^t & 0 & -\frac{1}{d}B_{d,n} \\ 0 & \cdots & \cdots & 0 & B_{d,n}^t & 0 \end{pmatrix}$$

**Proposition 5.1.** *The distribution of the eigenvalues of  $Q_{d,n}$  converges to  $\nu$  as  $n \rightarrow \infty$  such that  $\int -\ln(1-x)\nu(dx) = \frac{\ln d}{d-1}$ .*

*Proof.* a) In order to prove this, we need to study  $\det(1 - \rho Q_{d,n+1})$ . During the calculation the determinant, we meet the following sequences  $(a_n, n \geq 0)$  and  $(b_n, n \geq 0)$  defined as follows (the relation between the determinant and the sequences will be showed in part b) of the proof): For all  $\rho \in [0, 1]$ , let  $a_0(\rho) = 1$ ,  $a_1(\rho) = 1 - \frac{(d-1)\rho^2}{d}$ . For  $n \geq 1$ , let  $a_{n+1}(\rho) = 1 - \frac{(d-1)\rho^2}{d^2 a_n}$  and  $b_{n+1}(\rho) = 1 - \frac{\rho^2}{d a_n}$ .

It can be recurrently showed that  $a_i \in [1/d, 1]$  for  $i \in \mathbb{N}$ :

First,  $a_0, a_1 \in [1/d, 1]$ . If  $a_n \in [1/d, 1]$ , then  $a_{n+1} < 1$  and  $a_{n+1} \geq 1 - \frac{(d-1)\rho^2}{d} \geq 1 - \frac{d-1}{d} = 1/d$ . Therefore,  $a_i \in [1/d, 1]$  for  $i \in \mathbb{N}$ .

Moreover, one could show that  $(a_n, n \geq 1)$  is non-decreasing with limit  $\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{(d-1)\rho^2}{d^2}}$ : consider  $f(x) = 1 - \frac{(d-1)\rho^2}{d^2 x}$ ,  $x > 0$ . Then  $a_n = f^n(1/d)$ .  $f$  is increasing on  $]0, \infty[$  and

$$\begin{cases} f(x) < x & x \in ]0, \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{(d-1)\rho^2}{d^2}}[ \cup ]\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{(d-1)\rho^2}{d^2}}, \infty[ \\ f(x) \geq x & x \in ]\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{(d-1)\rho^2}{d^2}}, \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{(d-1)\rho^2}{d^2}}[ \end{cases}$$

Since  $f(1/d) \geq 1/d$ , one has  $\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{(d-1)\rho^2}{d^2}} \leq 1/d \leq \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{(d-1)\rho^2}{d^2}}$ . Consequently,  $(a_n, n \geq 1)$  is non-decreasing with limit  $\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{(d-1)\rho^2}{d^2}}$ .

b) Next, we will prove that for  $\rho \in [0, 1]$ ,

$$-\ln \det(1 - \rho Q_{d,n+1})/m_{n+1} = -\frac{\ln b_{n+1}(\rho) + d \sum_{k=0}^n (d-1)^k \ln a_{n-k}(\rho)}{1 + d \sum_{k=0}^n (d-1)^k}$$

Since  $m_{n+1} = 1 + d \sum_{k=0}^n (d-1)^k$ , it is enough to show

$$\det(1 - \rho Q_{d,n+1}) = b_{n+1} \prod_{i=0}^n a_i(\rho)^{d(d-1)^{n-i}}$$

From the expression of  $Q_{d,n+1}$  for  $n \geq 2$ , we obtain

$$I - \rho Q_{d,n+1} = \begin{pmatrix} 1 & -\frac{\rho}{d}E_{1,d} & 0 & \cdots & \cdots & 0 \\ -\frac{\rho}{d}E_{1,d}^t & I_d & -\frac{\rho}{d}B_{d,1} & \ddots & \ddots & \vdots \\ 0 & -\frac{\rho}{d}B_{d,1}^t & I_{d(d-1)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\frac{\rho}{d}B_{d,n-1} & 0 \\ \vdots & \ddots & \ddots & -\frac{\rho}{d}B_{d,n-1}^t & I_{d(d-1)^{n-1}} & -\frac{\rho}{d}B_{d,n} \\ 0 & \cdots & \cdots & 0 & -\rho B_{d,n}^t & I_{d(d-1)^n} \end{pmatrix}$$

Suppose  $A$  is a  $m \times m$  matrix,  $B$  is a  $m \times n$  matrix,  $C$  is a  $n \times m$  matrix and  $D$  is a  $n \times n$  revertible matrix, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \det(D)$$

Write  $I - \rho Q_{d,n+1}$  as  $\begin{pmatrix} A & B \\ C & I_{d(d-1)^n} \end{pmatrix}$ , then use the formula above, to obtain

$$\begin{aligned} & \det(I - \rho Q_{d,n+1}) \\ &= \det \begin{pmatrix} 1 & -\frac{1}{d}E_{1,d} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{d}E_{d,1} & I_d & -\frac{\rho}{d}B_{d,1} & \ddots & \ddots & \vdots \\ 0 & -\frac{\rho}{d}B_{d,1}^t & I_{d(d-1)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\frac{\rho}{d}B_{d,n-2} & 0 \\ \vdots & \ddots & \ddots & -\frac{\rho}{d}B_{d,n-2}^t & I_{d(d-1)^{n-2}} & -\frac{\rho}{d}B_{d,n-1} \\ 0 & \cdots & \cdots & 0 & -\frac{\rho}{d}B_{d,n-1}^t & I_{d(d-1)^{n-1}} - \frac{\rho^2}{d}B_{d,n}B_{d,n}^t \end{pmatrix} \end{aligned}$$

Since  $B_{d,n}B_{d,n}^t = (d-1)I_{d(d-1)^{n-1}}$ ,

$$\det(I - \rho Q_{d,n+1}) = \det \begin{pmatrix} 1 & -\frac{1}{d}E_{1,d} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{d}E_{d,1} & I_d & -\frac{\rho}{d}B_{d,1} & \ddots & \ddots & \vdots \\ 0 & -\frac{\rho}{d}B_{d,1}^t & I_{d(d-1)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\frac{\rho}{d}B_{d,n-2} & 0 \\ \vdots & \ddots & \ddots & -\frac{\rho}{d}B_{d,n-2}^t & I_{d(d-1)^{n-2}} & -\frac{\rho}{d}B_{d,n-1} \\ 0 & \cdots & \cdots & 0 & -\frac{\rho}{d}B_{d,n-1}^t & a_1(\rho)I_{d(d-1)^{n-1}} \end{pmatrix}$$

We do this repeatedly,

$$\begin{aligned}
& \det(I - \rho Q_{d,n+1}) \\
&= a_1(\rho)^{d(d-1)^{n-1}} \det \begin{pmatrix} 1 & -\frac{1}{d}E_{1,d} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{d}E_{d,1} & I_d & -\frac{\rho}{d}B_{d,1} & \ddots & \ddots & \vdots \\ 0 & -\frac{\rho}{d}B_{d,1}^t & I_{d(d-1)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\frac{\rho}{d}B_{d,n-3} & 0 \\ \vdots & \ddots & \ddots & -\frac{\rho}{d}B_{d,n-3}^t & I_{d(d-1)^{n-3}} & -\frac{\rho}{d}B_{d,n-2} \\ 0 & \cdots & \cdots & 0 & -\frac{\rho}{d}B_{d,n-2}^t & a_2(\rho)I_{d(d-1)^{n-2}} \end{pmatrix} \\
&= \cdots = a_1(\rho)^{d(d-1)^{n-1}} \cdots a_{n-1}(\rho)^{d(d-1)} \det \begin{pmatrix} 1 & -\frac{1}{d}E_{1,d} \\ -\frac{1}{d}E_{d,1} & a_n(\rho)I_d \end{pmatrix} \\
&= b_{n+1} \prod_{i=1}^n a_i(\rho)^{d(d-1)^{n-i}} \\
&= b_{n+1} \prod_{i=0}^n a_i(\rho)^{d(d-1)^{n-i}}
\end{aligned}$$

c) The limit as  $n \rightarrow \infty$ :

For  $\rho \in [0, 1[$  and  $d \geq 3$ , let  $n$  tend to infinity,

$$\lim_{n \rightarrow \infty} -\ln(1 - \rho Q_{d,n})/m_n = -\frac{d-2}{d-1} \sum_{k \geq 1} (d-1)^{-k} \ln a_k(\rho)$$

By Proposition 3.1, the distribution of the eigenvalues of  $Q_{d,n}$  converges. Denote the limit distribution by  $\nu$ . Since  $\lim_{\rho \rightarrow 1} a_k(\rho) = 1/d$ ,  $\int -\ln(1-x)\nu(dx) = \lim_{\rho \rightarrow 1} \lim_{n \rightarrow \infty} -\ln(1 - \rho Q_{d,n})/m_n = \frac{\ln d}{d-1}$  for  $d \geq 3$ . For  $d = 2$ , the argument is similar,

$$\lim_{n \rightarrow \infty} -\ln(1 - \rho Q_{2,n})/m_n = -\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \ln a_i(\rho)}{n} = -\lim_{n \rightarrow \infty} \ln a_n(\rho) = -\ln \frac{1 + \sqrt{1 - \rho^2}}{2}$$

The distribution of the eigenvalues of  $Q_{2,n}$  converges to  $\nu$  such that  $\int -\ln(1-x)\nu(dx) = \ln 2$ .  $\square$

Finally, for  $d \geq 3$ :

- a.1) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{(d-1)^n} \leq 0$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 0$ .
- b) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{(d-1)^n} = \infty$ ,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 1$ .
- c) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{(d-1)^n} = a \in ]0, \infty[$ ,

$$\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = \frac{a^{\frac{d-2}{d}}}{\frac{\ln d}{d-1} + a^{\frac{d-2}{d}}}$$

While for  $d = 2$ :

- a.1) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{2n} \leq 0$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 0$ .
- b) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{2n} = \infty$ ,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 1$ .
- c) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{2n} = a \in ]0, \infty[$ ,

$$\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = \frac{a}{\ln 2 + a}$$

The result is the same as the 1-dimensional discrete torus. It can be viewed as an example of Proposition 3.2.

## 6 The case of the complete graph

Let  $(G_n = (V_n, E_n, w_n))$  be the complete graph of  $n$ -vertices with equal edge weight 1. Reconsider the same problem and use the same notation as before. The problem is to find the limit of  $\mathfrak{P}_n(\mathcal{C})$ . Since there is no universal degree bound, the result could not be derived from Theorem 2.7.

**Lemma 6.1.**

$$\text{tr} Q_n^k = \frac{(n-1)(-1)^k + (n-1)^k}{(n-1)^k}$$

Fix some  $x \in V_n$ ,

$$\text{tr}(Q_n|_{\{V_n - \{x\}\}})^k = \frac{(n-2)(-1)^k + (n-2)^k}{(n-1)^k}$$

*Proof.* Let  $(X_m^{(n)}, m \geq 0)$  be the Markov chain on  $V_n$  with transition matrix  $Q_n$ . Then

$tr Q_n^k = \sum_{y \in V_n} \mathbb{P}^y[X_k^{(n)} = y] = n \mathbb{P}^y[X_k^{(n)} = y]$  for any  $y \in V_n$ . For  $k \geq 1$ ,

$$\begin{aligned} tr Q_n^k &= n \mathbb{P}^y[X_k^{(n)} = y] = n \sum_{z \neq y} \mathbb{P}^y[X_{k-1}^{(n)} = z] Q_y^z = \frac{n}{n-1} \sum_{z \neq y} \mathbb{P}^y[X_{k-1}^{(n)} = z] \\ &= \frac{n}{n-1} (1 - \mathbb{P}^y[X_{k-1}^{(n)} = y]) = \frac{n}{n-1} - \frac{tr Q_n^{k-1}}{n-1} \end{aligned}$$

Therefore,  $(n-1)^k tr Q_n^k = -(n-1)^{k-1} tr Q_n^{k-1} + n(n-1)^{k-1}$ . Consequently,

$$tr Q_n^k = \frac{(n-1)(-1)^k + (n-1)^k}{(n-1)^k}$$

Since  $Q_n|_{\{V_n - \{x\}\}} = \frac{n-2}{n-1} Q_{n-1}$ ,

$$tr(Q_n|_{\{V_n - \{x\}\}})^k = \frac{(n-2)(-1)^k + (n-2)^k}{(n-1)^k}$$

□

**Proposition 6.2.** *For any  $\epsilon > 0$  fixed,  $\mathfrak{P}_n(\mathcal{C} | p(l) = k)$  tends to 1 uniformly for  $k \geq n^{1+\epsilon}$  as  $n$  tends to infinity, i.e.*

$$\lim_{n \rightarrow \infty} \sup_{k \geq n^{1+\epsilon}} |1 - \mathfrak{P}_n(\mathcal{C} | p(l) = k)| = 0$$

*Proof.* Fix some vertex  $x$ ,

$$\begin{aligned} &\mathfrak{P}_n(1 \text{ does not cover the vertex } x | p(l) = k) \\ &= \frac{\mu_n(1 \text{ does not cover the vertex } x, p(l) = k)}{\mu_n(p(l) = k)} \\ &= \frac{tr(Q_n|_{V_n - \{x\}})^k}{tr Q_n^k} \\ &= \frac{(n-2)(-1)^k + (n-2)^k}{(n-1)(-1)^k + (n-1)^k} \end{aligned}$$

Therefore,  $\mathfrak{P}_n(\mathcal{C} | p(l) = k) \geq 1 - \sum_{x \in V_n} \mathfrak{P}_n(1 \text{ does not cover the vertex } x | p(l) = k) = 1 - n \frac{(n-2)(-1)^k + (n-2)^k}{(n-1)(-1)^k + (n-1)^k}$ . Consequently,

$$\lim_{n \rightarrow \infty} \sup_{k \geq n^{1+\epsilon}} |1 - \mathfrak{P}_n(\mathcal{C} | p(l) = k)| = 0$$

□

**Proposition 6.3.** a) For  $k > 1$ ,  $\mu_n(p(l) = k) = \frac{1}{k}(\frac{1}{1+c_n})^k(1 + (-\frac{1}{n-1})^{k-1}) = \frac{1}{k}(\frac{1}{1+c_n})^k(1 + o(\frac{1}{n}))$

b) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} < 1$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) < n) = 1$ .

c) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = \infty$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) > n^2) = 1$ .

d) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = d \in [1, \infty[$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) < n) = 1/d$  and  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(n \leq p(l) < n^{1+\epsilon}) \leq \epsilon/d$ .

*Proof.* a) For  $k > 1$ ,  $\mu_n(p(l) = k) = \frac{1}{k}(\frac{1}{1+c_n})^k \text{tr} Q_n^k = \frac{1}{k}(\frac{1}{1+c_n})^k(1 + (-\frac{1}{n-1})^{k-1}) = \frac{1}{k}(\frac{1}{1+c_n})^k(1 + o(\frac{1}{n}))$

b) Suppose  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} < 1$ . Then  $\exists \epsilon > 0, N > 0, \forall n \geq N, c_n > n^{\epsilon-1}$ .

$\mu_n(p(l) < n) \geq \mu_n(p(l) = 2) \sim \frac{1}{2} \frac{1}{(1+c_n)^2}$  as  $n \rightarrow \infty$ . And  $\lim_{n \rightarrow \infty} \mu_n(p(l) \geq n) \leq \lim_{n \rightarrow \infty} \sum_{k \geq n} (\frac{1}{1+c_n})^k = \lim_{n \rightarrow \infty} \frac{1}{c_n(1+c_n)^{n-1}} = o(\mu_n(p(l) < n))$ . Therefore,

$$\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) < n) = \lim_{n \rightarrow \infty} \frac{\mu_n(p(l) < n)}{\mu_n(p(l) < n) + \mu_n(p(l) \geq n)} = 1$$

c) Suppose  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = \infty$ . Then  $\forall A > 0, \exists N > 0, \forall n \geq N, c_n \leq n^{-A}$ .

$$\mu_n(2 \leq p(l) \leq n^2) = \sum_{k=2}^{n^2} \frac{1}{k} (\frac{1}{1+c_n})^k (1 + (-\frac{1}{n-1})^{k-1}) \leq 2 \sum_{k=2}^{n^2} \frac{1}{k} \leq 4 \ln n = o(-\ln c_n)$$

and  $\mu_n(p(l) \geq 2) = \sum_{k=2}^{\infty} \frac{1}{k} (\frac{1}{1+c_n})^k (1 + (-\frac{1}{n-1})^{k-1}) \sim \sum_{k=2}^{\infty} \frac{1}{k} \frac{1}{(1+c_n)^k} \sim -\ln c_n$ . Therefore,

$$\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) > n^2) = 1.$$

d.1) Suppose  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = d \in ]1, \infty[$ . Then  $\exists \epsilon > 0, N > 0$  such that  $\forall n \geq N, c_n \leq n^{-(1+\epsilon)}$ .

Since  $\lim_{n \rightarrow \infty} \sup_{k \leq n} |\frac{1}{(1+c_n)^k} - 1| = 0$ ,  $\mu_n(p(l) < n) \sim \sum_{k=2}^n \frac{1}{k} \sim \ln n$ . And  $\mu_n(p(l) \geq 2) \sim \sum_{k=2}^{\infty} \frac{1}{k} \frac{1}{(1+c_n)^k} \sim -\ln c_n$ . Therefore,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) < n) = 1/d$ . Since  $\lim_{n \rightarrow \infty} \mu_n(n \leq p(l) \leq n^{1+\epsilon}) \leq \sum_{k=n}^{n^{1+\epsilon}} \frac{1}{k} \sim \epsilon \ln n$ ,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(n \leq p(l) < n^{1+\epsilon}) \leq \epsilon/d$ .

d.2) Suppose  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = 1$ . Then  $\forall \epsilon > 0, \exists N > 0$  such that  $\forall n \geq N, c_n \leq n^{-(1-\epsilon)}$ .

$$\mu_n(p(l) \geq 2) \sim \sum_{k=2}^{\infty} \frac{1}{k} \frac{1}{(1+c_n)^k} \sim -\ln c_n \sim -\ln n$$

$$\mu_n(p(l) \in [2, n^{1-2\epsilon}]) \sim \sum_{k=2}^{n^{1-2\epsilon}} \frac{1}{k} \sim (1-2\epsilon) \ln n$$



Therefore,  $\forall \epsilon > 0$ ,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) \in [2, n]) \geq \lim_{n \rightarrow \infty} \frac{\mu_n(p(l) \in [2, n^{1-2\epsilon}])}{\mu_n(p(l) \geq 2)} \geq 1 - 2\epsilon$ . Consequently,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(p(l) \in [2, n]) = 1$ . Since  $\lim_{n \rightarrow \infty} \mu_n(n \leq p(l) \leq n^{1+\epsilon}) \leq \sum_{k=n}^{n^{1+\epsilon}} \frac{1}{k} \sim \epsilon \ln n$ ,  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(n \leq p(l) < n^{1+\epsilon}) \leq \epsilon$ .  $\square$

Combining the above properties, one has the following theorem,

**Theorem 6.4.** *a) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} \leq 1$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 0$ .  
b) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = \infty$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 1$ .  
c) If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = d \in ]1, \infty[$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 1 - 1/d$ .*

**Corollary 4.** *Let  $\mathcal{L}_n$  be the Poisson collection of loops with intensity  $\frac{\mu_n}{\ln n}$ . Suppose  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = d$ . Then,  $\sum_{l \in \mathcal{L}_n} 1_{\{l \in \mathcal{C}\}}$  converges in distribution to a Poisson random variable with parameter  $(d - 1)_+$  as  $n$  tends to infinity. In the case  $|V_n| = m_n$ , replace  $\frac{-\ln c_n}{\ln n}$  by  $\frac{-\ln c_n}{\ln m_n}$ .*

*Proof.* By Proposition 6.3, the total mass of  $\mu_n$ , namely  $||\mu_n||$ , is equivalent to  $\sum_{k \geq 2} \frac{1}{k} \left(\frac{1}{1+c_n}\right)^k = -\ln(1 - \frac{1}{1+c_n}) - \frac{1}{1+c_n}$ . Using the result of Theorem 6.4, if  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} \leq 1$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 0$ , i.e.  $\mu_n(\mathcal{C}) = o(||\mu_n||) = o(\ln n)$ . Therefore,  $\lim_{n \rightarrow \infty} \frac{\mu_n(\mathcal{C})}{\ln n} = 0$ . If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = \infty$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 1$ , i.e.  $\mu_n(\mathcal{C}) \sim ||\mu_n|| \sim -\ln c_n$ . Therefore,  $\lim_{n \rightarrow \infty} \frac{\mu_n(\mathcal{C})}{\ln n} = \infty$ . If  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = d \in ]1, \infty[$ , then  $\lim_{n \rightarrow \infty} \mathfrak{P}_n(\mathcal{C}) = 1 - 1/d$ , i.e.  $\mu_n(\mathcal{C}) = (1 - 1/d)||\mu_n|| \sim -(1 - 1/d) \ln c_n \sim (d - 1) \ln n$ . Therefore,  $\lim_{n \rightarrow \infty} \frac{\mu_n(\mathcal{C})}{\ln n} = d - 1$ . Combining the argument above together, suppose  $\lim_{n \rightarrow \infty} \frac{-\ln c_n}{\ln n} = d$ , then  $\lim_{n \rightarrow \infty} \frac{\mu_n(\mathcal{C})}{\ln n} = (d - 1)_+$ . Consequently,  $\sum_{l \in \mathcal{L}_n} 1_{\{l \in \mathcal{C}\}}$  converges in distribution to a Poisson random variable with parameter  $(d - 1)_+$  as  $n$  tends to infinity.  $\square$

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